10 Predictability Basics

10.1 A Model of Convecting Fluids: The Lorenz Model

Essentially from the textbook: Chaos and Nonlinear Dynamics. Robert C. Hilborn, Oxford University Press, 1994.

The set of nonlinear equations (discuss linear oscillator, and exponential decay!) derived in this section is a highly simplyfied model of a convecting fluid. The model was introduced in 1963 by MIT meteorologist Edward Lorenz, who was interested in modelling convection in the atmosphere. What Lorenz set out to demonstrate was that even a very simple set of equations may have solutions whose behaviour is essentially unpredictable. Unfortunately for the development of the science of chaos, Lorenz published his results in the respectable, but little read Journal of the Atmospheric Sciences, where they languished essentially unnoticed by mathematicians and scientists in other fields until the 1970s (about 10 citations until 1972, then 3000 in one year.......). Now that chaos is more widely appreciated, a minor industry studying the Lorenz model equations has developed.

Here, the Lorenz equations will not be derived, we will just say enough to give a feeling for what the equations tell us. In simple physical terms, the Lorenz model treats the fluid system (say the atmosphere) as a fluid layer that is heated at the bottom (due to the sun's heating the earth's surface, for example) and cooled at the top. The bottom of the fluid is maintained at a temperature T_w (the 'warm' temperature), which is higher than the temperature T_c (the 'cold' temperature) at the top. We will assume that the temperature difference $T_w - T_c$ is held fixed. (This type of system was studied experimentally by Benard in 1900. Lord Rayleigh provided a theoretical understanding of some basic features in 1916. Hence, this configuration is now called *Rayleigh-Benard* cell.) Fig. 68 shows the principle set-up.

If the temperature difference $\delta T = T_w - T_c$ is not too large, the fluid will remain stationary. Heat is transferred from bottom to top by means of thermal conduction. The tendency of warm (less dense) fluid to rise is counterbalanced by a loss of heat from the warm fluid 'packet' to the surrounding medium. The damping due to the fluid viscosity prevents the packet from rising more rapidly than the time required for it to come to the same temperature as its neighbours. Under these conditions the temperature drops *linearly* with the vertical position from T_w at the bottom of the layer to T_c at the top. However, if the temperature difference becomes large enough, the buoyant forces eventually become strong enough to overcome viscosity and steady circulation currents develop. In this situation heat is transferred from the bottom to the top by the process of convection, the actual mass motion of the fluid. In simple terms, when the warm packet of fluid reaches the top of the layer, it looses heat to the cool region and then sinks to the bottom, where its temperature goes up again. The net result is a circulation pattern that is stable in time.

With a further increase in temperature difference δT, the circulation currents and the resulting temperature differences within the fluid start to vary in time.

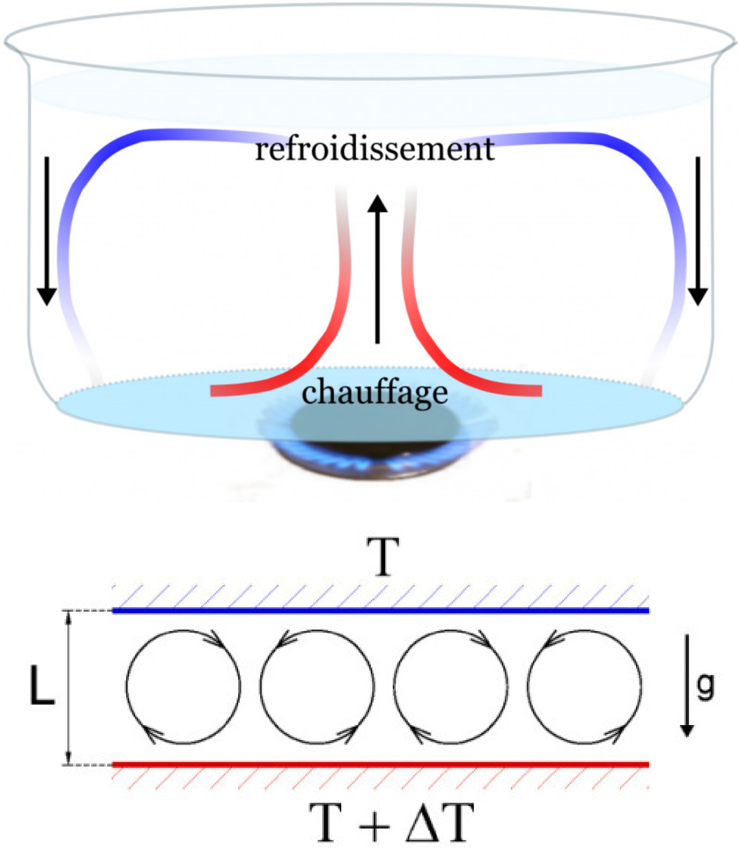


Figure 68: Setting of Rayleigh-Benard convection.

This never occurs for a linear system with frictional forces included. If a linear system is subject to steady forces, (after an initial transition period) will be steady in time.

10.2 The Lorenz Equations

The Lorenz model is based on a gross simplification of the fundamental Navier-Stokes equations for fluids (explain the approach to insert fixed spatial dependencies of solutions into the Navier-Stokes equations, then look for solutions of the time-dependent amplitudes). The fluid motion and resulting temperature differences can be expressed in terms of these three variables, conventionally called $X(t), Y(t)$ and $Z(t)$. These are *not* spatial variables. X is related to the time-dependence of the so-called fluid stream function. The variables Y and Z are related to the time dependence of the temperature deviations away from the linear temperature drop from bottom to top, which one obtains for the nonconvective steady-state situation. In particular, Y is proportional to the temperature difference between the rising and falling parts of the fluid at a given height, while Z is proportional to the deviation from temperature linearity as a function of vertical position.

Using these variables, we may write the Lorenz model equations as three coupled differential equation

$$\begin{aligned} \dot{X} &= p(Y - X) \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ \ . \end{aligned} \tag{163}$$

p, r, b are adjustable parameters: p is the so-called Prandtl number, which is defined to be the ratio of kinetic viscosity of the fluid to its thermal diffusion coefficient. r is proportional to the *Rayleigh number*, which is a dimensionless measure of the temperature difference between the bottom and top of the fluid. As the temperature difference increases, the Rayleigh number increases. The final parameter b is related to the ratio of the vertical height h of the fluid layer to the horizontal size of the convection rolls. It turns out that for $b = 8/3$, the convection begins for the smallest value of the Rayleigh number, that is for the smallest value of the temperature difference δT. This value is usually chosen to study the Lorenz model. p is then chosen for the particular fluid under study. Lorenz (LOR63) used the value $p = 10$ (which corresponds roughly to cold water), a value that had been used in a previous study of Rayleigh-Benard convection by Saltzman (SAL62). We let r, the Rayleigh number, be the adjustable control parameter. The Lorenz model, although based on what appears to be a very simple set of differential equations, exhibits very complex behaviour. The equations look so simple that one is led to guess that it would be easy to write down their solutions. In fact, it is now believed that it is in principle impossible to give the solutions in analytical form. Thus, we must solve the equations numerically (explain possible discretization!). Here we will discuss a few results of those integrations.

10.3 Behaviour of Solutions to the Lorenz Equations

For small values of the parameter r, that is, for small temperature differences, δT, the model predicts that the stationary, nonconvecting state is the stable condition. In terms of the variables X, Y, Z, this state is described by the values $X_{s1} = 0, Y_{s1} = 0, Z_{s1} = 0$. For values of r greater that 1, steady convection sets in (is is actually quite easy to perform a linear stability analysis of this stationary point at zero, and it is left to you as an exercise, if you wish.....). There are two possible convective states: one corresponding to clockwise rotation, the other to counterclockwise rotation (discuss calculation of stationary points: $X_{s2,3} = \pm\sqrt{b(r-1)}$, $Y_{s2,3} = \pm\sqrt{b(r-1)}$, $Z_{s2,3} = r - 1$, solutions 2,3 only exist for $r > 1$). Some initial conditions lead to one state, other initial conditions to the other state. For $p > b + 1$, this steady convection is *unstable* for large enough r and gives way to more complex behaviour. As r increases, the behaviour has regions of chaotic behaviour intermixed with regions of periodic behaviour and regions of *intermittency*, which cycle back and forth, apparently random, between chaotic and periodic behaviour. Solutions are shown in Fig. 69 for different values of r (discuss dependence of solutions on initial conditions for all cases).

Note that even though the Lorenz equations strictly only apply to the Rayleigh-Benard convection experiment (and even here they are crude approximations to the full equations), these equations are applied to study the behaviour of many complex systems. For example, there is a vast literature on application to Indian monsoon intraseasonal variability, extratropical flow regimes, etc.

10.4 Kinds of predictability

It is the dependence of the solutions on small variations (uncertainties) in the initial conditions of our nonlinear system (Navier-Stokes equations), that leads us to the introduction of the concepts of predictability. Adrian Tompkins will in his lecture on Numerical Weather Prediction (NWP) discuss what the sources of uncertainties in the initial conditions are (obviousely related to measurement errors). There are other sources of uncertainties related to model imperfections that may also be included in the treatment, but are excluded here at the moment.

From Lorenz system we can understand the two principle *kinds of predictabilities*. The first kind is the (atmospheric) initial condition predictability, which can be illustrated looking at the development of an initial condition ensemble (this means many different initial conditions that have to be considered because of our uncertainties in the initial conditions). Fig. 70 illustrates the time development of such an initial condition ensemble in the Lorenz system. Predictability that results from the atmospheric initial conditions is the subject of NWP, and the limit turns out to be a few days. The NWP problem will be discussed by Adrian Tompkins in depth.

The more general case also including model uncertainty is illustrated in Fig. 71, which again illustrates the principle of growth in time of the differences between the different simulations, which could be interpreted as measure of uncertainty.

The predictability is usually measured using the *ensemble* method, meaning that several realizations of a prediction are performed from only slightly different initial conditions.

Fig. 72 shows the mean (root-mean-sqare) difference of the near surface air temperature at one gridpoint in Europe (upper panel) and equatorial Africa (lower panel) from the ECMWF seasonal hindcast ensemble (15 members) for slightly different initial conditions (we will discuss later what exactly this system is). As can be seen the error grows quickly in the first few days (even faster for equatorial Africa), then saturates more or less around 15-20 days for Europe, but already after 2-3 days in equatorial Africa, and at a much smaller saturation value. This feature has important implications for seasonal predictability in extratropical and tropical regions!

Usually, initial condition predictability of the atmosphere (particularly in the extratropics) is limited by a typical *error growth* time scale of a few days, which effectively means that no reasonable prediction can be made after a week or so (see

Fig. 72). This initial condition predictability is also referred to as *predictability of the first kind*. The *predictability of the second kind* results from the fact that the solutions of nonlinear systems (like e.g. the Lorenz system) stay in any case relatively close to the unstable stationary points, or *Attractors*. If any *external forcing* changes these stationary points, the solutions will change systematically with the changing stationary points. We can imagine that such a situation occurs, for example, in the atmosphere in the case of a large *El Nino* event. The atmospheric weather or *noise* will be chaotic and unpredictable on a seasonal time-scale (see Fig. 72), but the (imaginary) stationary state around which the atmosphere evolves may be shifting far enough that we can predict the average climate in this situation. This is what is referred to as predictability of the second kind (for the atmosphere). This is the usual situation we consider in *seasonal forecasts*. It is interesting for this to consider the differences in the saturation errors seen in Fig. 72, which is much larger in extratropical regions compared with tropical regions. This indicates potentially more predictability of the second kind in the tropics, even if predictability of the first kind is less. We will see in the following sections that this is indeed generally the case.

Note, however, that the separation between predictability of first and second kind is a little arbitrary and we need to specify exactly for which system. In all above reference to predicability of first and second kind I have added *atmosphere*. Indeed, seasonal forecasts are performed with coupled ocean-atmosphere models. For the ocean-atmospheric system, we are looking again at the initial condition predictability, or predictability of the first kind even for seasonal forecasts. The predictability relies then on the fact that some parts of the oceans evolve on much longer time-scales compared to the atmosphere (e.g. ENSO has a period of 2-7 years!). The same argument may apply to other 'external' forcing, for example, carbon dioxide may be considered as external forcing in many standard models, but if our model includes a carbon cycle, then it is part of the dynamical system itself. As a note, some researchers have also argued that an external forcing may not necessarily shift the attractor, but may express itself as a change of probability of the system to be close to one or the other stationary state (see Fig 73 from paper by Corti, Molteni and Palmer), which may make it difficult to identify, e.g. Climate change from observations.

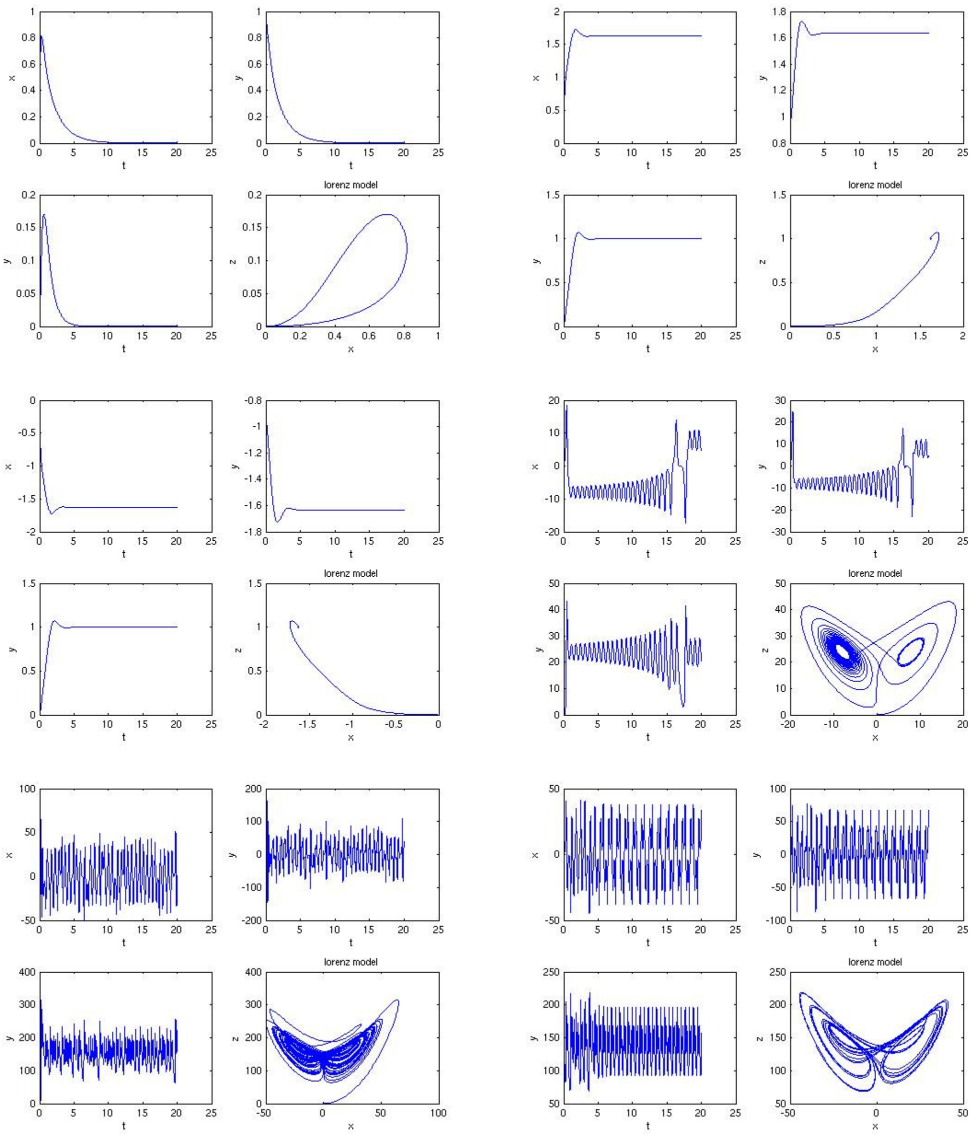


Figure 69: Solutions of the Lorenz equations for different parameters r.

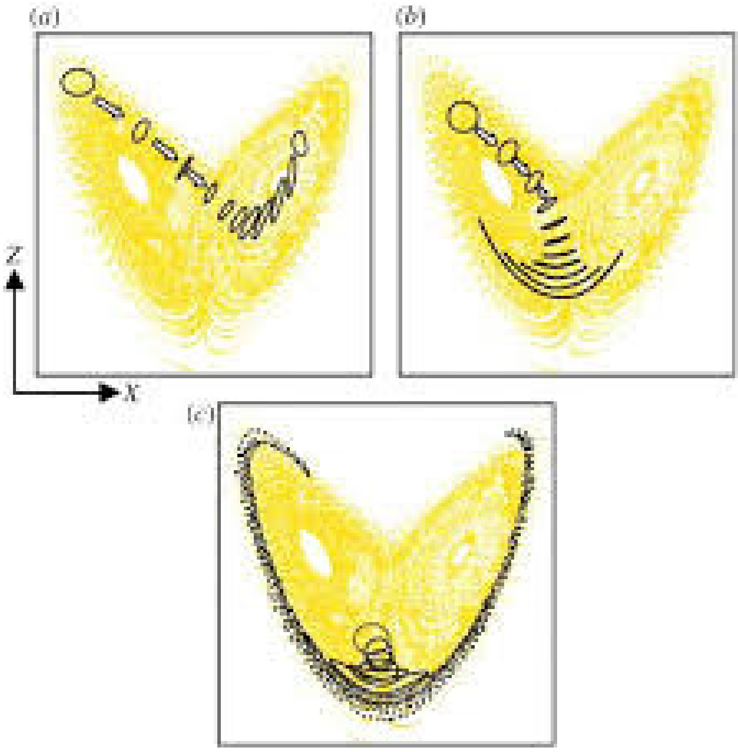


Figure 70: Illustration of time evolution of an initial condition ensemble in the Lorenz system.

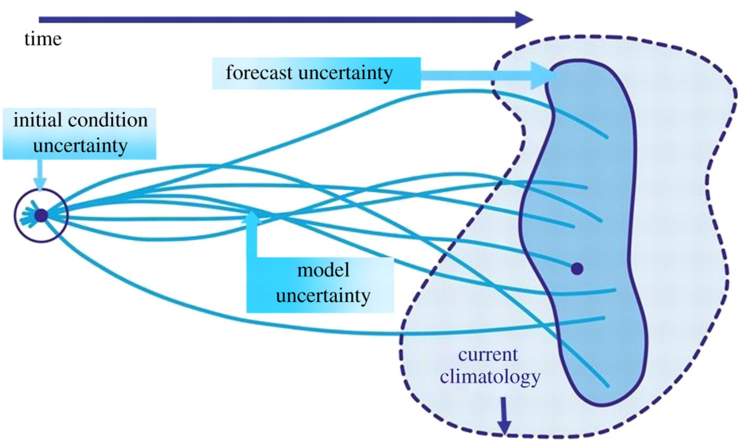
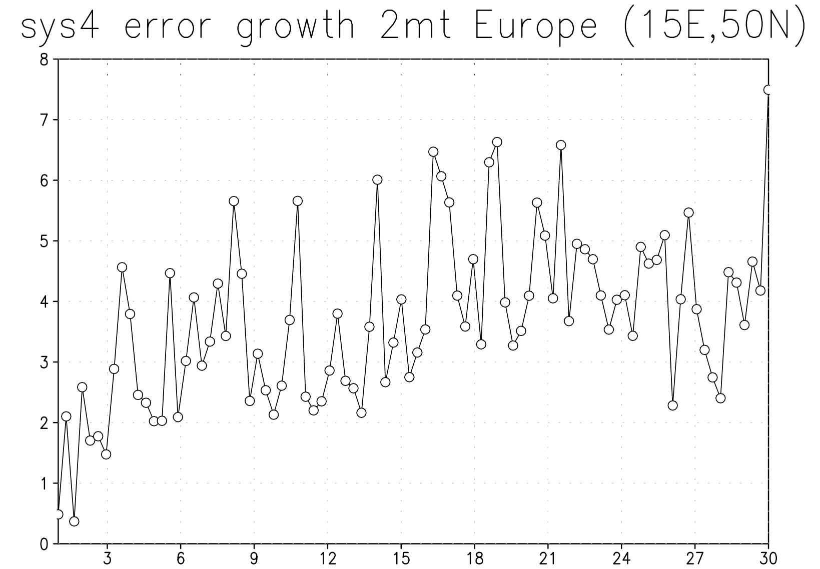


Figure 71: Illustration of the predictability problem.



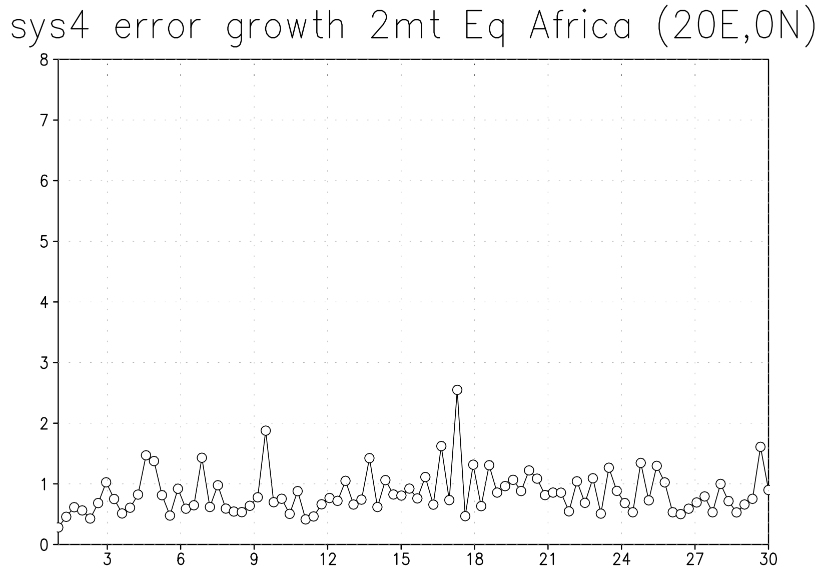


Figure 72: The mean (root-mean-square) difference of the near surface air temperatureat one gridpoint in Europe (15E, 50N) and equatorial Africa (20E, 0N) from the ECMWF seasonal hindcast ensemble (15 members) for slightly different initial conditions. Units are K.

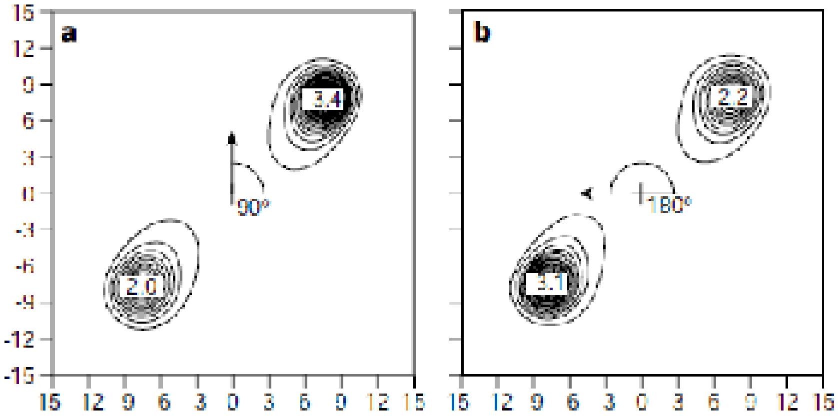


Figure 1 Response of a nonlinear chaotic model to imposed forcing. Illustrated is the state vector PDF (invariant measure) of a forced Lorenz[4] attractor, with governing equations $\dot{X} = -10X + 10Y + 2.5\cos\theta$, $\dot{Y} = -XZ + 28X - Y + 2.5\sin\theta$, $\dot{Z} = XY - (8/3)Z$, in X, Y phase space. The arrow shown is the forcing vector $(2.5\cos\theta, 2.5\sin\theta)$ in this two-dimensional phase space. A short running time-average has been applied to the state vector to emphasise the regime character of the Lorenz attractor. **a**, $\theta = 90°$; **b**, $\theta = 180°$. The numbers shown correspond to maxima of the PDF at the regime centroids. The quantities plotted on the horizontal and vertical axes are values of X and Y respectively. See ref. 3 for details.

Figure 73: Illustration of frequency of occurence changes in a Lorenz system from paper Corti, S. , F. Molteni and T. N. Palmer, 1999: Signature of recent climate change in frequencies of natural atmospheric circulation regimes. *Nature*, **398**, 799-802